Materials discussed on 1/11

We have discussed the following.

1. Show that
$$\zeta^*(z) = \frac{z}{z-1} - z\sum_{n=1}^{\infty}\int_n^{n+1} \frac{x-n}{x^{z+1}}dx$$
define a meromorphic function on $\{z : Re(z) > 0\}$ with pole at $z = 1$ and agrees with $\zeta(z)$ on $\{Re(z) > 1\}$.

2. Show that for all z where $Re(z) \in (-1, 0)$,
$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin(\frac{\pi z}{2}).$$

Proof. To make thing more rigorous, we extend ζ step by step first.

Claim: $\zeta(z)\Gamma(z) = \int_0^{\infty} \frac{t^{z-1}}{e^t - 1}\, dt$**.**

Noted that we only evaluate at the region in which the convergence is uniform. And hence there will be no issue on the interchange of limiting process.

$$\begin{aligned}
\zeta(z)\Gamma(z) &= (\sum_{n=1}^{\infty} n^{-z})\int_0^{\infty} e^{-t}t^{z-1}\, dt \\
&= \sum_{n=1}^{\infty}\int_0^{\infty} e^{-t}n^{-z}t^{z-1}\, dt \\
&= \sum_{n=1}^{\infty}\int_0^{\infty} e^{-nt}t^{z-1}\, dt \\
&= \int_0^{\infty} \frac{t^{z-1}}{e^t-1}\, dt, \qquad \forall\, z, Re(z) > 1.
\end{aligned}$$

Claim: $\zeta(z)$ **can be analytically extended to** $\{z : Re(z) > 0\}$**.**

Noted that $\Gamma(z)$ is meromorphic on $\mathbb{C}$. It suffices to show that the Right hand side above can be extended. From the integral representation, we can observe that t^{z-1} is integrable for $Re(z) > 0$. Thus, the only issues arise from $(e^t - 1)^{-1}$. From this perspective, we try to extract its singularity. Considering its power series,
$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} a_n z^n.$$

This suggests us to splits it into
$$\int_0^{\infty} \frac{t^{z-1}}{e^t-1}\, dt = \int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}\, dt + \frac{1}{z-1} + \int_1^{\infty} \frac{t^{z-1}}{e^t-1}\, dt.$$

By Morera's theorem, the integrability is sufficient to argue the analyticity. For sake of completeness, let me demonstrate the argument here.

Denote $f(z) = -\frac{1}{2} + \sum_{n=1}^{\infty} a_n z^n$ which is clearly holomorophic function around 0. (More precisely, it is entire.) We now show that $\int_0^1 f(t)t^{z-1}\,dt$ defines a holomorphic function on $\{z : Re(z) > \epsilon\}$ for any $\epsilon > 0$.

Clearly, $\int_{n^{-1}}^1 f(t)t^{z-1}\,dt$ is holomorphic and it is convergent when $n \to \infty$. It suffices to show that the convergence is uniform locally.

$$\begin{aligned}
\left| \int_{n^{-1}}^1 f(t)t^{z-1}\,dt - \int_{m^{-1}}^1 f(t)t^{z-1}\,dt \right| &\leq \int_{m^{-1}}^{n^{-1}} |f(t)||t^{z-1}|\,dt \\
&\leq C \int_{m^{-1}}^{n^{-1}} t^{Re(z)-1}\,dt \\
&\leq C \frac{t^{Re(z)}}{Re(z)}\Big|_{m^{-1}}^{n^{-1}} \\
&\leq \frac{C}{\epsilon}\frac{1}{n^{Re(z)}},
\end{aligned}$$

where C depends on the local maximum of f. Hence we have extended the $\zeta(z)$ to $\{Re(z) > 0\}$.

Claim: $\zeta(z)$ **can be analytically extended to** $\{z : Re(z) > -1\}$**.**

We consider $0 < Re(z) < 1$ first.

$$\begin{aligned}
\int_1^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\,dt &= \int_1^{\infty} \frac{1}{e^t - 1} t^{z-1}\,dt - \int_1^{\infty} t^{z-2}\,dt \\
&= \int_1^{\infty} \frac{1}{e^t - 1} t^{z-1}\,dt + \frac{1}{z-1}
\end{aligned}$$

Hence, if $Re(z) \in (0,1)$,

$$\begin{aligned}
\int_0^{\infty} \frac{t^{z-1}}{e^t - 1}\,dt &= \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\,dt + \int_1^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\,dt \\
&= \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\,dt + \int_1^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\,dt - \frac{1}{2z}.
\end{aligned}$$

Now we examine each integral.

The first one define a holomorphic function on $\{Re(z) > -1\}$ by observing $\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} = O(t)$ around $t = 0$.

On the other hand, the second integral also defines a holomorphic function on $\{Re(z) < 1\}$ as we have

$$|\frac{1}{e^t - 1} - \frac{1}{t}| \leq \frac{C}{t}$$

for some $C > 0$, for all $t \geq 1$. From this point of view, we can extend ζ to $\{Re(z) > -1\}$.

Now if $-1 < Re(z) < 0$, since $\int_1^\infty t^{z-1} dt = -z^{-1}$,

$$\int_0^\infty \frac{t^{z-1}}{e^t - 1}\, dt = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} dt. \quad = \int_0^\infty \left(\frac{i}{2} \cot(it/2) - \frac{1}{t} \right) t^{z-1}\, dt$$

Then we can use the formula for $\cot(z) = \frac{1}{z} + \sum_{n=1}^\infty \frac{2z}{z^2 - n^2\pi^2}$ to recover the functional equation. The rest is direct computation. Try this! $\square$